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# The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere

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## Abstract

We present the gravity dual to a class of three-dimensional  $\mathcal{N} = 2$  supersymmetric gauge theories on a biaxially squashed three-sphere, with a non-trivial background gauge field. This is described by a 1/2 BPS Euclidean solution of four-dimensional  $\mathcal{N} = 2$  gauged supergravity, consisting of a Taub-NUT-AdS metric with a non-trivial instanton for the graviphoton field. The holographic free energy of this solution agrees precisely with the large  $N$  limit of the free energy obtained from the localized partition function of a class of Chern-Simons quiver gauge theories. We also discuss a different supersymmetric solution, whose boundary is a biaxially squashed Lens space  $S^3/\mathbb{Z}_2$  with a topologically non-trivial background gauge field. This metric is of Eguchi-Hanson-AdS type, although it is not Einstein, and has a single unit of gauge field flux through the  $S^2$  cycle.

# 1 Introduction

Supersymmetric gauge theories on compact curved backgrounds are interesting for various reasons. For example, supersymmetry may be combined with localization techniques, allowing one to perform a variety of exact computations in strongly coupled field theories. The authors of [1] presented a construction of  $\mathcal{N} = 2$  supersymmetric gauge theories in three dimensions in the background of a  $U(1) \times U(1)$ -invariant squashed three-sphere and R-symmetry gauge field. The gravity dual of this construction was recently given in [2]. It consists of a 1/4 BPS Euclidean solution of four-dimensional  $\mathcal{N} = 2$  gauged supergravity, which in turn may be uplifted to a supersymmetric solution of eleven-dimensional supergravity. In particular, the bulk metric in [2] is simply  $\text{AdS}_4$ , and the graviphoton field is an instanton with (anti)-self-dual field strength. The asymptotic metric and gauge field then reduce to the background considered in [1].

The purpose of this letter is to present the gravity dual to a different field theory construction, obtained recently in [3]. In this reference the authors have constructed three-dimensional  $\mathcal{N} = 2$  supersymmetric gauge theories in the background of the  $SU(2) \times U(1)$ -invariant squashed three-sphere (which we refer to as *biaxially* squashed) and a non-trivial background  $U(1)$  gauge field, and have computed the corresponding partition functions using localization. Differently from a similar construction discussed briefly in [1], this partition function depends non-trivially on the squashing parameter. As we will see, the gravity dual to this set-up will have some distinct features with respect to the solution in [2]. In particular, the metric is not simply  $\text{AdS}_4$ , although it will again be an Einstein metric, and there is a self-dual graviphoton.

The plan of the rest of this paper is as follows. In section 2 we review the construction of [3]. In section 3 we discuss the gravity dual. In section 4 we describe a different supersymmetric solution, consisting of a non-Einstein metric and a non-instantonic graviphoton field. Section 5 concludes.

## 2 Supersymmetric gauge theories on the biaxially squashed $S^3$

In the construction of [3] the metric on the three-sphere is, up to an irrelevant overall factor, given by

$$ds_3^2 = \sigma_1^2 + \sigma_2^2 + \frac{1}{v^2} \sigma_3^2, \quad (2.1)$$

where  $\sigma_i$  are the standard  $SU(2)$  left-invariant one-forms on  $S^3$ , defined as  $i\sigma_i\tau_i = -2\mathbf{g}^{-1}d\mathbf{g}$ , where  $\tau_i$  denote the Pauli matrices and  $\mathbf{g} \in SU(2)$ . The background  $U(1)$  gauge field reads

$$A^{(3)} = \frac{\sqrt{v^2 - 1}}{2v^2} \sigma_3 , \quad (2.2)$$

and the spinors in the supersymmetry transformations obey the equation (setting the radius  $r = 2$  in [3])

$$\nabla_\alpha^{(3)} \chi - \frac{i}{4v} \gamma_\alpha \chi - A_\beta^{(3)} \gamma_\alpha{}^\beta \chi = 0 , \quad (2.3)$$

where  $\nabla_\alpha^{(3)}$ ,  $\alpha = 1, 2, 3$ , is the spinor covariant derivative constructed from the metric (2.1), and  $\gamma_\alpha$  generate  $\text{Cliff}(3, 0)$ . There are *two* linearly independent solutions to (2.3), transforming as a doublet under  $SU(2)$ , whose explicit form is given in [3]. This will be important for identifying the gravity dual.

In [3] the authors constructed Chern-Simons, Yang-Mills, and matter Lagrangians for the  $\mathcal{N} = 2$  vector multiplets  $V = (\mathcal{A}_\alpha, \sigma, \lambda, D)$  and chiral multiplets  $\Phi = (\phi, \psi, F)$ , in the background of the metric (2.1) and R-symmetry gauge field (2.2). These are invariant under a set of supersymmetry transformations, provided the spinorial parameters obey the equation (2.3). The supersymmetric completion of the Chern-Simons Lagrangian contains new terms, in addition to those appearing in flat space, proportional to  $\sigma^2$  and  $\sigma A^{(3)} \wedge d\mathcal{A}$  (*cf.* eq. (32) of [3]). The Yang-Mills and matter Lagrangians are total supersymmetry variations (*cf.* eq. (31) of [3]) and therefore can be used to compute the partition function using localization. In particular, the partition function localizes on supersymmetric configurations obeying

$$\mathcal{A}_\alpha = D = 0 , \quad \sigma = u = \text{constant} , \quad (2.4)$$

with the matter fields all being zero. Notice that although  $D = 0$ , the Chern-Simons Lagrangian is non-zero because of the new term proportional to  $\sigma^2$ , and therefore it contributes classically to the localized partition function, as in previous constructions. The Yang-Mills and matter terms contribute one-loop determinants from the Gaussian integration about the the classical solutions (2.4). The final partition function may be expressed again in terms of double sine functions  $s_b(z)$ , and for a  $U(N)$  gauge theory at Chern-Simons level  $k \in \mathbb{Z}$  reads

$$Z = \int \prod_{\text{Cartan}} du \exp \left( \frac{i\pi k}{v^2} \text{Tr } u^2 \right) \frac{\prod_{\text{Roots } \alpha} s_b \left( \frac{\alpha(u) - i}{v} \right)}{\prod_{\text{Chirals, rep } \mathcal{R}_a} \prod_{\rho \in \mathcal{R}_a} s_b \left( \frac{\rho(u) - i(1 - \Delta_a)}{v} \right)} , \quad (2.5)$$

where  $b = (1 + i\sqrt{v^2 - 1})/v$ . The exponential term is the classical contribution from the Chern-Simons Lagrangian, evaluated on (2.4); the numerator is the one-loop vector multiplet determinant and involves a product over the roots  $\alpha$  of the gauge group  $G$ ; while the denominator is the one-loop matter determinant and involves a product over chiral fields of R-charge  $\Delta_a$  in representations  $\mathcal{R}_a$ , with  $\rho$  running over weights in the weight-space decomposition of  $\mathcal{R}_a$ . Following [4], one can easily extract the large  $N$  behaviour of this partition function for a class of non-chiral  $\mathcal{N} = 2$  quiver Chern-Simons-matter theories. The calculation was done in [3], and the result is that the leading contribution to the free energy (defined as  $\mathcal{F} = -\log Z$ ) is given by

$$\mathcal{F}_v = \frac{1}{v^2} \mathcal{F}_{v=1} , \quad (2.6)$$

and thus depends very simply on the squashing parameter  $v$ . In the next section we will present the supergravity dual to this construction, in particular showing that the holographic free energy precisely agrees with the field theory result (2.6).

### 3 The gravity dual

As anticipated in [2], we will show that the gravity dual to the set-up described in the previous section is a supersymmetric solution of  $d = 4$ ,  $\mathcal{N} = 2$  gauged supergravity. In *Lorentzian* signature, the bosonic part of the action is given by

$$S_{\text{Lorentzian}} = \frac{1}{16\pi G_4} \int d^4x \sqrt{-\det g_{\mu\nu}} [R + 6g^2 - (F^L)^2] . \quad (3.1)$$

Here  $R$  denotes the Ricci scalar of the four-dimensional metric  $g_{\mu\nu}$ , and the cosmological constant is given by  $\Lambda = -3g^2$ . The graviphoton is an Abelian gauge field  $A^L$  with field strength  $F^L = dA^L$ ; here the superscript  $L$  emphasizes that this is a Lorentzian signature object. A solution to the equations of motion derived from (3.1) is supersymmetric if there is a non-trivial spinor  $\epsilon$  satisfying the Killing spinor equation

$$[\nabla_\mu + \tfrac{1}{2}g\Gamma_\mu - igA_\mu^L + \tfrac{i}{4}F_{\nu\rho}^L\Gamma^{\nu\rho}\Gamma_\mu]\epsilon = 0 . \quad (3.2)$$

Here  $\Gamma_\mu$ ,  $\mu = 0, 1, 2, 3$ , generate the Clifford algebra  $\text{Cliff}(1, 3)$ , so  $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$ .

Since the background of [3] preserves half of the maximal supersymmetry in three dimensions, we should seek a  $1/2$  *BPS Euclidean* solution of  $d = 4$ ,  $\mathcal{N} = 2$  gauged supergravity, whose metric has as conformal boundary the biaxially squashed metric on  $S^3$  (2.1), and whose background  $U(1)$  gauge field restricted to this asymptotic

boundary reduces to (2.2). This very strongly suggests that the appropriate solution is a Euclideanized version of the 1/2 BPS Reissner-Nordström-Taub-NUT-AdS solution discussed in [5].

We will first present this Euclidean solution, and then discuss the Wick rotation that leads to it. The metric reads

$$ds_4^2 = \frac{r^2 - s^2}{\Omega(r)} dr^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2\Omega(r)}{r^2 - s^2} \sigma_3^2 , \quad (3.3)$$

where

$$\Omega(r) = (s - r)^2 [1 + g^2(r - s)(r + 3s)] , \quad (3.4)$$

and  $s$  is the NUT parameter.<sup>1</sup> The  $SU(2)$  left-invariant one-forms  $\sigma_i$  may be written in terms of angular variables as

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\varphi) , \quad \sigma_3 = d\psi + \cos\theta d\varphi . \quad (3.5)$$

The graviphoton field is

$$A = s \frac{r - s}{r + s} \sqrt{1 - 4g^2 s^2} \sigma_3 . \quad (3.6)$$

In the orthonormal frame

$$\begin{aligned} e_1 &= \sqrt{r^2 - s^2} \sigma_1 , & e_2 &= \sqrt{r^2 - s^2} \sigma_2 , \\ e_3 &= 2s \sqrt{\frac{\Omega(r)}{r^2 - s^2}} \sigma_3 , & e_4 &= \sqrt{\frac{r^2 - s^2}{\Omega(r)}} dr , \end{aligned} \quad (3.7)$$

the curvature may be written as

$$F = dA = -\frac{s\sqrt{1 - 4g^2 s^2}}{(r + s)^2} (e_{12} + e_{34}) . \quad (3.8)$$

Thus the gauge field is an *instanton*, as in the solution discussed in [2]. In particular, with our choice of orientation the curvature is self-dual, and the on-shell gauge field action is finite. Since the stress-energy tensor of an instanton vanishes, the metric (3.3) is accordingly an Einstein metric. However, differently from the solution in [2], one can check that this metric is *not* locally  $\text{AdS}_4$ . It is in fact a Euclidean version of the well-known Taub-NUT-AdS metric, with a special value of the mass parameter. This metric is locally asymptotically  $\text{AdS}_4$ , and therefore it can be interpreted holographically [6].

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<sup>1</sup>This is denoted  $N$  in [5].

Notice that for  $|s| \leq 1/(2g)$  the gauge field (3.6) is *real*, while for  $|s| > 1/(2g)$  it is *purely imaginary*; the intermediate case with  $|s| = 1/(2g)$  has vanishing gauge field instanton and the metric reduces to Euclidean  $\text{AdS}_4$ .

For large  $r$  the metric becomes

$$ds_4^2 \approx \frac{dr^2}{g^2 r^2} + r^2 [\sigma_1^2 + \sigma_2^2 + 4g^2 s^2 \sigma_3^2] , \quad (3.9)$$

while to leading order the gauge field reduces to

$$A \approx A^{(3)} \equiv s \sqrt{1 - 4g^2 s^2} \sigma_3 . \quad (3.10)$$

We see that the conformal boundary may be identified *precisely* with the metric (2.1), and the background gauge field with (2.2), by setting  $s = \frac{1}{2gv}$ . Recall here that in order to uplift to eleven-dimensional supergravity one should also set  $g = 1$  [2]. Notice that when  $|v| = 1$  the boundary metric reduces to the round metric on  $S^3$ , and the background gauge field vanishes. Correspondingly, in the bulk the instanton field vanishes, and the metric becomes  $\text{AdS}_4$ .

## Wick rotation and regularity

Let us discuss briefly how this solution was obtained. The reader not interested in these details may safely jump to the discussion of the Killing spinors and the holographic free energy.

As we are interested in a 1/2 BPS solution, we may begin by appropriately Wick rotating the solution (2.1), (2.4) of [5]. We take their parameter  $\aleph = +1$ , so as to obtain a biaxially squashed  $S^3$  as constant  $r$  surface. The Wick rotation may then be taken to be  $t \rightarrow i\tau$ ,  $N \rightarrow is$ ,  $Q \rightarrow iQ$ , together with a change in sign of the metric. This leads to the following metric and gauge field

$$\begin{aligned} ds_4^2 &= \frac{r^2 - s^2}{\Omega(r)} dr^2 + (r^2 - s^2)(d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{\Omega(r)}{r^2 - s^2} (d\tau + 2s \cos \theta d\varphi)^2 , \\ A^L &= \frac{sP - Qr}{r^2 - s^2} d\tau + \frac{P(r^2 + s^2) - 2sQr}{r^2 - s^2} \cos \theta d\varphi , \end{aligned} \quad (3.11)$$

where

$$\Omega(r) = g^2(r^2 - s^2)^2 + (1 - 4g^2 s^2)(r^2 + s^2) - 2Mr + (P^2 - Q^2) . \quad (3.12)$$

This depends on the parameters  $s, g, M, P, Q$ . Notice we have kept a Lorentzian superscript on  $A^L$  in (3.11) – the reason for this will become clear momentarily.

For the 1/2 BPS solution of interest, the Euclideanized BPS equations of [5] imply that

$$\begin{aligned} M^2 &= (1 - 4g^2s^2) [s^2(1 - 4g^2s^2) + P^2 - Q^2] , \\ s^2P(1 - 4g^2s^2) &= sMQ - P(P^2 - Q^2) , \end{aligned} \quad (3.13)$$

and the corresponding 1/2 BPS solution then depends on only *two* parameters. We take these to be  $s$  and  $Q$ , with

$$P = is\sqrt{1 - 4g^2s^2} , \quad M = -iQ\sqrt{1 - 4g^2s^2} , \quad (3.14)$$

then solving (3.13). The factors of  $i$  in (3.14) may look problematic, but there are (at least) two different ways of obtaining real solutions. We require  $s$  and  $M$  to be real in order that the metric in (3.11) is real. If  $|s| \leq 1/(2g)$  then  $P$  and  $Q$  will be purely imaginary, and we may write  $P = ip$ ,  $Q = -iq$  to obtain the *real* gauge field

$$A \equiv -iA^L = \frac{sp + qr}{r^2 - s^2} d\tau + \frac{p(r^2 + s^2) + 2sqr}{r^2 - s^2} \cos\theta d\varphi . \quad (3.15)$$

Redefining  $\tau = 2s\psi$ , in terms of standard Euler angles  $(\theta, \varphi, \psi)$  notice that the metric (3.11) takes the form presented in (3.3), albeit with a more general form of the function  $\Omega(r)$ , given by (3.12) and (3.14). That (3.3) has only one free parameter  $s$ , and not the two we have above, follows from imposing *regularity* of the Euclidean metric. At any fixed  $r > s$  that is not a root of  $\Omega(r)$ , we obtain a smooth biaxially squashed  $S^3$  metric. In order to obtain a complete metric, the space must “close off” at the largest root  $r_0$  of  $\Omega(r)$ , so that  $\Omega(r_0) = 0$ . More precisely, if  $r_0 > s$  this should be a single root, while if  $r_0 = s$  the metric will be regular only if  $r_0 = s$  is a double root of  $\Omega(r)$ . We shall return to the former case in section 4, here focussing on the case  $r_0 = s$ . The condition  $\Omega(r_0 = s) = 0$  immediately fixes

$$q = -s\sqrt{1 - 4g^2s^2} , \quad (3.16)$$

so that now (see also [7])

$$p = s\sqrt{1 - 4g^2s^2} = -q , \quad M = s(1 - 4g^2s^2) . \quad (3.17)$$

It is then in fact automatic that  $r = s$  is a double root of  $\Omega$ .

In conclusion, we end up with the metric (3.3), with  $\Omega(r)$  given in (3.4), and gauge field (3.6). The gauge field is manifestly non-singular and one can check that the metric indeed smoothly closes off at  $r = s$ , giving the topology  $M_4 = \mathbb{R}^4$ .



## Killing spinors

In this subsection we briefly discuss the supersymmetry of the Euclidean solution (3.3), (3.6), in particular reproducing the three-dimensional spinor equation (2.3) asymptotically.

In Lorentzian signature the Killing spinor equation is (3.2). However, in Wick rotating we have introduced a factor of  $i$  into the gauge field in (3.15), so that  $A^L = iA$ . Thus the appropriate Killing spinor equation to solve in this case is

$$[\nabla_\mu + \frac{1}{2}g\Gamma_\mu + gA_\mu - \frac{1}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu]\epsilon = 0. \quad (3.18)$$

This possibility of Wick rotating the gauge field (or not) was also discussed in [8]. In particular, the authors of [8] pointed out that any Euclidean solution with a real gauge field that solves (3.18) will automatically be 1/2 BPS. The reason is simple: if  $\epsilon$  solves (3.18), then so does its conjugate  $\epsilon^c$ . We shall see this explicitly below.

We introduce the following representation for the generators of  $\text{Cliff}(4, 0)$

$$\hat{\Gamma}_4 = \begin{pmatrix} 0 & i\mathbb{I}_2 \\ -i\mathbb{I}_2 & 0 \end{pmatrix}, \quad \hat{\Gamma}_\alpha = \begin{pmatrix} 0 & \tau_\alpha \\ \tau_\alpha & 0 \end{pmatrix}, \quad (3.19)$$

where  $\alpha \in 1, 2, 3$ ,  $\tau_\alpha$  are the Pauli matrices, and hats denote tangent space quantities. Decomposing the Dirac spinor  $\epsilon$  into positive and negative chirality parts as

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}, \quad (3.20)$$

where  $\epsilon_\pm$  are two-component spinors, it is then straightforward, but tedious, to verify that in the orthonormal frame (3.7)

$$\epsilon_+ = \begin{pmatrix} \lambda(r)\chi_+ \\ \lambda^*(r)\chi_- \end{pmatrix}, \quad \epsilon_- = i\sqrt{\frac{r-s}{r+s}} \begin{pmatrix} \lambda^*(r)\chi_+ \\ \lambda(r)\chi_- \end{pmatrix}, \quad (3.21)$$

is the general solution to the  $\mu = r$  component of (3.18), where  $\chi_\pm$  are independent of  $r$  and we have defined

$$\lambda(r) \equiv \left(g(r+s) - i\sqrt{1-4g^2s^2}\right)^{1/2}. \quad (3.22)$$

If we now define the charge conjugate spinor  $\epsilon^c \equiv B\epsilon^*$ , where  $B$  is the charge conjugation matrix defined in [2], then it is straightforward to see that taking the conjugate  $\epsilon \rightarrow \epsilon^c$  simply maps  $\chi_+ \rightarrow -\chi_-^*$ ,  $\chi_- \rightarrow \chi_+^*$ .

Let us analyze the large  $r$  asymptotics of the Killing spinor equation (3.18), and its solutions (3.21). We begin by expanding

$$\begin{aligned}\epsilon_+ &= \sqrt{g}r^{1/2} \left[ \mathbb{I}_2 + \left( \frac{s}{2}\mathbb{I}_2 - \frac{i}{2g}\sqrt{1-4g^2s^2\tau_3} \right) r^{-1} + \mathcal{O}(r^{-2}) \right] \chi , \\ \epsilon_- &= i\sqrt{g}r^{1/2} \left[ \mathbb{I}_2 - \left( \frac{s}{2}\mathbb{I}_2 - \frac{i}{2g}\sqrt{1-4g^2s^2\tau_3} \right) r^{-1} + \mathcal{O}(r^{-2}) \right] \chi ,\end{aligned}\quad (3.23)$$

where we have defined the  $r$ -independent two-component spinor

$$\chi \equiv \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} . \quad (3.24)$$

We then write the asymptotic expansion of the metric as

$$ds_4^2 = \frac{dr^2}{g^2r^2} [1 + \mathcal{O}(r^{-2})] + \frac{r^2}{g^2} [ds_3^2 + \mathcal{O}(r^{-2})] , \quad (3.25)$$

$$ds_3^2 \equiv g^2 [\sigma_1^2 + \sigma_2^2 + 4g^2s^2\sigma_3^2] . \quad (3.26)$$

It is then straightforward to extract the coefficient of  $r^{1/2}$  in the Killing spinor equation (3.18). One finds that the positive and negative chirality projections lead to the *same* equation for  $\chi$ , namely

$$\nabla_\alpha^{(3)}\chi + gA_\alpha^{(3)}\chi - \frac{is}{2}\gamma_\alpha\chi - \frac{1}{2g}\sqrt{1-4g^2s^2}\gamma_\alpha\tau_3\chi = 0 , \quad (3.27)$$

where  $\nabla^{(3)}$  denotes the spin connection for the three-metric (3.26), and  $A^{(3)}$  is defined in (3.10). Using the explicit form for  $A^{(3)}$  in (3.10), the identity  $\gamma_\alpha\gamma_\beta = \gamma_{\alpha\beta} + g_{\alpha\beta}^{(3)}$ , and recalling that  $s = 1/(2gv)$ ,  $g = 1$ , we precisely obtain the spinor equation (2.3). Finally, one can verify that the  $d = 4$  spinors (3.21), with  $\chi$  satisfying (3.27), do indeed solve (3.18).

## The holographic free energy

The holographic free energy of the Taub-NUT-AdS solution was discussed in [7], but of course in this latter reference there was no instanton field, which is crucial for supersymmetry. The calculation proceeds essentially as in section 2.5 of [2], except for the following caveat. The integrability condition for the Killing spinor equation (3.18) gives the equations of motion following from the action

$$S_{\text{Euclidean}} = -\frac{1}{16\pi G_4} \int d^4x \sqrt{\det g_{\mu\nu}} (R + 6g^2 + F^2) , \quad (3.28)$$

which has *opposite* (relative) sign for the gauge field term compared with (3.1) (see also [8]). This is clear from the fact that our equation (3.18) was obtained from the Lorentzian form of the equation by sending  $A \rightarrow iA$ . It is therefore natural to expect that in the computation of the holographic free energy we have to evaluate the action  $S_{\text{Euclidean}}$  on shell.

Setting  $g = 1$  and cutting off the space at  $r = R$ , the bulk gravity contribution is given by

$$I_{\text{bulk}}^{\text{grav}} = \frac{3}{8\pi G_4} \int d^4x \sqrt{\det g_{\mu\nu}} = \frac{4\pi s R^3}{G_4} - \frac{12\pi s^3 R}{G_4} + \frac{8\pi s^4}{G_4}. \quad (3.29)$$

Denoting by  $R[\gamma]$  the scalar curvature of the boundary metric, and by  $K$  the trace of its second fundamental form, the combined gravitational boundary terms

$$I_{\text{ct}}^{\text{grav}} + I_{\text{bdry}}^{\text{grav}} = \frac{1}{8\pi G_4} \int d^3x \sqrt{\det \gamma_{\alpha\beta}} \left( 2 + \frac{1}{2} R[\gamma] - K \right) \quad (3.30)$$

have the following asymptotic expansion

$$I_{\text{ct}}^{\text{grav}} + I_{\text{bdry}}^{\text{grav}} = -\frac{4\pi s R^3}{G_4} + \frac{12\pi s^3 R}{G_4} + \frac{4\pi s^2(1 - 4s^2)}{G_4} + \mathcal{O}(1/R), \quad (3.31)$$

where in particular notice there is a non-zero *finite* contribution. The instanton action is<sup>2</sup>

$$I_{\text{bulk}}^F = -\frac{1}{16\pi G_4} \int d^4x \sqrt{\det g_{\mu\nu}} F_{\mu\nu} F^{\mu\nu} = -\frac{2\pi s^2(1 - 4s^2)}{G_4}. \quad (3.32)$$

Therefore the total on-shell action  $S_{\text{Euclidean}}$ , obtained after removing the cut-off ( $R \rightarrow \infty$ ), is given by

$$I = I_{\text{bulk}}^{\text{grav}} + I_{\text{ct}}^{\text{grav}} + I_{\text{bdry}}^{\text{grav}} + I_{\text{bulk}}^F = \frac{2\pi s^2}{G_4}. \quad (3.33)$$

Since the round sphere result<sup>3</sup> is  $s = 1/2$ , we thus see that

$$I_s = \frac{2\pi s^2}{G_4} = (2s)^2 I_{s=1/2}, \quad (3.34)$$

which since  $v = 1/(2s)$  precisely agrees with the field theory result (2.6).

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<sup>2</sup>Notice that when  $1 - 4s^2 \geq 0$  this term becomes negative. The calculation is however valid for any value of  $s > 0$ .

<sup>3</sup>To recover the result for  $S^2 \times S^1$  boundary, one should first change coordinates back to the form in (3.11), and then set  $s = 0$  there. In these coordinates, with  $\tau \in [0, 2\pi]$  the gravitational contribution to the free energy is half that of the round sphere.

## 4 A supersymmetric Eguchi-Hanson-AdS solution

In the previous section the Taub-NUT-AdS solution existed for both  $1 - 4g^2s^2 \geq 0$  and  $1 - 4g^2s^2 \leq 0$ , with the sign determining whether the gauge field is real or purely imaginary, in a fixed choice of Wick rotation. However, in this section we consider a different solution which exists only when  $1 - 4g^2s^2 \leq 0$ , or equivalently  $|s| \geq 1/(2g)$ . In this case the Euclidean supersymmetry equation takes the same form as the Lorentzian equation (3.2), namely

$$[\nabla_\mu + \frac{1}{2}g\Gamma_\mu - igA_\mu + \frac{i}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu]\epsilon = 0. \quad (4.1)$$

We will show that there is a one-parameter family of regular solutions in this class, of topology  $M_4 = T^*S^2$ , for which there are Killing spinors solving (4.1).

When  $|s| \geq 1/(2g)$  we may rewrite (3.14) as

$$P = -s\sqrt{4g^2s^2 - 1}, \quad M = Q\sqrt{4g^2s^2 - 1}, \quad (4.2)$$

which are now *real*. Again setting  $\tau = 2s\psi$ , the metric takes the form given in (3.3) where now

$$\Omega(r) = g^2(r^2 - s^2)^2 - \left[r\sqrt{4g^2s^2 - 1} + Q\right]^2. \quad (4.3)$$

It will be useful to note that the four roots of  $\Omega(r)$  in (4.3) are

$$\begin{aligned} \begin{Bmatrix} r_4 \\ r_3 \end{Bmatrix} &= \frac{1}{2g} \left[ \sqrt{4g^2s^2 - 1} \pm \sqrt{8g^2s^2 + 4gQ - 1} \right], \\ \begin{Bmatrix} r_2 \\ r_1 \end{Bmatrix} &= \frac{1}{2g} \left[ -\sqrt{4g^2s^2 - 1} \pm \sqrt{8g^2s^2 - 4gQ - 1} \right]. \end{aligned} \quad (4.4)$$

Notice that these are all complex if  $|s| < 1/(2g)$ . The gauge field is given by (after a suitable gauge transformation)

$$A = -\frac{s}{r^2 - s^2} \left[ 2Qr + (r^2 + s^2)\sqrt{4g^2s^2 - 1} \right] \sigma_3. \quad (4.5)$$

As  $r \rightarrow \infty$  this tends to

$$A \approx A^{(3)} \equiv -s\sqrt{4g^2s^2 - 1}\sigma_3, \quad (4.6)$$

which is (up to analytic continuation) what we had in the previous example (3.10).

## Killing spinors

Taking the same Clifford algebra and spinor conventions as the previous section, and again using the orthonormal frame (3.7), one can verify that the integrability condition for the Killing spinor equation (4.1) leads to the algebraic relation

$$\epsilon_- = i\sqrt{\frac{r-s}{r+s}} \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{(r-r_1)(r-r_2)}} & 0 \\ 0 & \sqrt{\frac{(r-r_1)(r-r_2)}{(r-r_3)(r-r_4)}} \end{pmatrix} \epsilon_+ . \quad (4.7)$$

Here recall that  $\epsilon_{\pm}$  are two-component spinors, and  $r_i$ ,  $i = 1, 2, 3, 4$ , are the four roots of  $\Omega$  in (4.4). Substituting into the  $\mu = r$  component of (4.1) then leads to decoupled first order ODEs, which may be solved to give

$$\epsilon_+ = \begin{pmatrix} \sqrt{\frac{(r-r_1)(r-r_2)}{(r-s)}} \chi_+ \\ \sqrt{\frac{(r-r_3)(r-r_4)}{(r-s)}} \chi_- \end{pmatrix}, \quad \epsilon_- = i \begin{pmatrix} \sqrt{\frac{(r-r_3)(r-r_4)}{(r+s)}} \chi_+ \\ \sqrt{\frac{(r-r_1)(r-r_2)}{(r+s)}} \chi_- \end{pmatrix}, \quad (4.8)$$

where  $\chi_{\pm}$  are independent of  $r$ . The large  $r$  expansion of these is given by

$$\epsilon_+ = r^{1/2} \left[ \mathbb{I}_2 + \left( \frac{s}{2} \mathbb{I}_2 + \frac{1}{2g} \sqrt{4g^2 s^2 - 1} \tau_3 \right) r^{-1} + \mathcal{O}(r^{-2}) \right] \chi, \quad (4.9)$$

$$\epsilon_- = ir^{1/2} \left[ \mathbb{I}_2 - \left( \frac{s}{2} \mathbb{I}_2 + \frac{1}{2g} \sqrt{4g^2 s^2 - 1} \tau_3 \right) r^{-1} + \mathcal{O}(r^{-2}) \right] \chi, \quad (4.10)$$

where the two-component spinor  $\chi$  is again given by (3.24). Notice this is the same as (3.23), up to analytic continuation. Again using the metric expansion and three-metric in (3.26), we may extract the coefficient of  $r^{1/2}$  in (4.1). A very similar computation to that in the previous section then leads to the three-dimensional Killing spinor equation

$$\nabla_{\alpha}^{(3)} \chi - \frac{is}{2} \gamma_{\alpha} \chi + ig A_{\beta}^{(3)} \gamma_{\alpha}^{\beta} \chi = 0. \quad (4.11)$$

Setting  $g = 1$  and again identifying the squashing parameter  $v = 1/(2s)$ , notice this is identical to our original equation (2.3), but where we have replaced  $A^{(3)} \rightarrow -iA^{(3)}$ . Of course, given the relative difference in Wick rotations of the gauge field in two the cases, this was precisely to be expected. In fact, comparing the  $A^{(3)}$  (4.6) in this section with its counterpart (3.10) in the previous section, we see that equation (4.11) is in fact *identical* to (2.3), due to the factor of  $i$  difference in (4.6), (3.10).

The solution to (4.11) is therefore given by an appropriate analytic continuation of the solution presented in [3], and reads

$$\chi = e^{\eta\tau_3/2} g^{-1} \chi_0, \quad (4.12)$$

where  $\mathbf{g} \in SU(2)$ ,  $\chi_0$  is a constant two-component spinor, and

$$v = \frac{1}{\cosh \eta} , \quad (4.13)$$

where  $v = 1/(2s)$ . In terms of Euler angles  $(\psi, \theta, \varphi)$ , recall that

$$\mathbf{g} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i(\psi+\varphi)/2} & \sin \frac{\theta}{2} e^{-i(\psi-\varphi)/2} \\ -\sin \frac{\theta}{2} e^{i(\psi-\varphi)/2} & \cos \frac{\theta}{2} e^{-i(\psi+\varphi)/2} \end{pmatrix} . \quad (4.14)$$

## Regularity of the metric

We must again consider regularity of the metric (3.3). A complete metric will necessarily close off at the largest root  $r_0$  of  $\Omega(r)$ , which must satisfy  $r_0 \geq s$ . From (4.4) we see that either  $r_0 = r_+$  or  $r_0 = r_-$ , where it is convenient to define

$$r_+ \equiv r_4 , \quad r_- \equiv r_2 . \quad (4.15)$$

*A priori* the coordinate  $\psi$  must have period  $2\pi/n$ , for some positive integer  $n$ , so that the surfaces of constant  $r$  are Lens spaces  $S^3/\mathbb{Z}_n$ . Assuming that  $r_0 > s$  is strict, then the metric (3.3) will have the topology of a complex line bundle  $M_4 = \mathcal{O}(-n) \rightarrow S^2$  over  $S^2$ , where  $r - r_0$  is the radial direction away from the zero section.

Regularity of the metric near to the  $S^2$  zero section at  $r = r_0$  requires

$$\left| \frac{r_0^2 - s^2}{s\Omega'(r_0)} \right| = \frac{2}{n} . \quad (4.16)$$

This condition ensures that near to  $r = r_0$  the metric (3.3) takes the form

$$ds_4^2 \approx d\rho^2 + \rho^2 \left[ d \left( \frac{n\psi}{2} \right) + \frac{n}{2} \cos \theta d\varphi \right]^2 + (r_0^2 - s^2)(d\theta^2 + \sin^2 \theta d\varphi^2) , \quad (4.17)$$

near to  $\rho = 0$ . Here note that  $n\psi/2$  has period  $2\pi$ . Imposing (4.16) at  $r_0 = r_{\pm}$  gives

$$Q = Q_{\pm}(s) \equiv \mp \frac{128g^4 s^4 - 16g^2 s^2 - n^2}{64g^3 s^2} . \quad (4.18)$$

In turn, substituting  $Q = Q_{\pm}(s)$  into (4.15) one then finds

$$r_{\pm}(Q_{\pm}(s)) = \frac{1}{8g} \left[ \frac{n}{gs} \pm 4\sqrt{4g^2 s^2 - 1} \right] . \quad (4.19)$$

Recall that in order to have a smooth metric, we require  $r_0 > s$ . Imposing this for  $r_0 = r_{\pm}(Q_{\pm}(s))$  gives

$$r_{\pm}(Q_{\pm}(s)) - s = \frac{1}{2g} f_n^{\pm}(2gs) , \quad (4.20)$$

where the function

$$f_n^\pm(x) \equiv \frac{n}{2x} - x \pm \sqrt{x^2 - 1} \quad (4.21)$$

is required to be *positive* for a smooth metric with  $s = x/(2g)$ . Notice here that  $s \geq 1/(2g)$  implies  $x \geq 1$ . It is straightforward to show that  $f_n^-(x)$  is monotonic decreasing on  $x \in [1, \infty)$ . For simplicity here we will restrict our attention to  $n \leq 2$ .<sup>4</sup> The analysis then splits into the cases  $\{n = 1\}$ ,  $\{n = 2\}$ , which have a qualitatively different behaviour:

$n = 1$

It is easy to see that  $f_1^\pm(x) < 0$  on  $x \in [1, \infty)$ , and thus the metric (3.3) cannot be made regular in this case. Specifically,  $f_1^\pm(1) = -1/2$ : since  $f_1^-(x)$  is monotonic decreasing, this rules out taking  $r_0 = r_-(Q_-(s))$  given by (4.19); on the other hand  $f_1^+(x)$  monotonically increases to zero from below as  $x \rightarrow \infty$ , and we thus also rule out  $r_0 = r_+(Q_+(s))$  in (4.19).

$n = 2$

It is easy to see that  $f_2^-(x) < 0$  for  $x \in (1, \infty)$ , while  $f_2^+(x) > 0$  on the same domain, which means we must set

$$Q \equiv Q_+(s) = -\frac{(4g^2s^2 - 1)(1 + 8g^2s^2)}{16g^3s^2}, \quad (4.22)$$

and

$$r_0(s) = \frac{1}{4g} \left[ \frac{1}{gs} + 2\sqrt{4g^2s^2 - 1} \right], \quad (4.23)$$

may then be shown to be the largest root of  $\Omega(r)$ , for all  $s \geq 1/(2g)$ . In particular, this involves showing that  $r_0(s) - r_-(Q_+(s)) > 0$  for all  $s \geq 1/(2g)$ , which follows since

$$r_0(s) - r_-(Q_+(s)) = \frac{1}{2g} h(2gs), \quad (4.24)$$

where we have defined

$$h(x) \equiv \frac{1}{x} + 2\sqrt{x^2 - 1} - \sqrt{4x^2 - 2 - \frac{1}{x^2}}. \quad (4.25)$$

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<sup>4</sup>In the first version of this paper it was argued that  $n > 2$  breaks supersymmetry; however, this is incorrect.

It is a simple exercise to prove that  $h(x) > 0$  on  $x \in (1, \infty)$ .

After this slightly involved analysis, for  $n = 2$  we end up with a smooth complete metric on  $M_4 = T^*S^2$ , given by (3.3), (4.3) with  $Q = Q_+(s)$  given by (4.22), for all  $s > 1/(2g)$ . The  $S^2$  zero section is at  $r = r_0(s)$  given by (4.23). The metric is thus of Eguchi-Hanson-AdS type, although we stress that it is *not* Einstein for any  $s > 1/(2g)$ . The large  $r$  behaviour is again given by (3.9), so that the conformal boundary is a squashed  $S^3/\mathbb{Z}_2$ . The  $s = 1/2g$  limit gives a round  $S^3/\mathbb{Z}_2$  at infinity with the bulk metric being the singular  $\text{AdS}_4/\mathbb{Z}_2$ , albeit with a non-trivial torsion gauge field, as we shall see momentarily.

It follows that another interesting difference to the Taub-NUT-AdS solution of the previous section is that the gauge field (4.5) no longer has (anti)-self-dual field strength  $F = dA$ ; moreover, the latter has a non-trivial flux. Indeed, although the gauge potential in (4.5) is *singular* on the  $S^2$  at  $r = r_0$ , one can easily see that the field strength  $F = dA$  is a globally defined smooth two-form on our manifold. One computes the period of this through the  $S^2$  at  $r_0(s)$  to be

$$\begin{aligned} \frac{g}{2\pi} \int_{S^2} F &= -\frac{2gs}{r_0(s)^2 - s^2} \left[ -2Q_+(s)r_0(s) - (r_0(s)^2 + s^2)\sqrt{4g^2s^2 - 1} \right] \\ &= 1, \end{aligned} \tag{4.26}$$

the last line simply being a remarkable identity satisfied by the largest root  $r_0(s)$ . Setting  $g = 1$ , we thus see that we have precisely one unit of flux through the  $S^2$ ! It follows that the gauge field  $A$  is a connection on the non-trivial line bundle  $\mathcal{O}(1) \rightarrow T^*S^2$ . The corresponding first Chern class  $c_1 = [F/2\pi] \in H^2(T^*S^2; \mathbb{Z}) \cong \mathbb{Z}$  is the generator of this group. Moreover, the map  $H^2(T^*S^2; \mathbb{Z}) \rightarrow H^2(S^3/\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2$  that restricts the gauge field to the conformal boundary is reduction modulo 2. Hence at infinity the background gauge field is more precisely given by the global one-form (4.6) *plus* the flat non-trivial Wilson line that represents the element  $1 \in H^2(S^3/\mathbb{Z}_2; \mathbb{Z}) \cong H_1(S^3/\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2$ . One would be able to see this explicitly by writing the gauge field  $A$  as a one-form that is locally well-defined in coordinate patches, and undergoes appropriate gauge transformations between these coordinate patches. It follows that the gauge field at infinity is more precisely a connection on the non-trivial torsion line bundle over  $S^3/\mathbb{Z}_2$ .



## The holographic free energy

Although we will not pursue the holographic interpretation of this solution in the present paper, below we will compute its holographic free energy using standard formulas. Since the gauge field here is real, the relevant action is the Euclidean action with standard signs

$$S_{\text{Euclidean}} = -\frac{1}{16\pi G_4} \int d^4x \sqrt{\det g_{\mu\nu}} (R + 6g^2 - F^2) . \quad (4.27)$$

Notice that upon taking the trace of the Einstein equation, we see that *all* solutions (supersymmetric or not) of  $d = 4$  gauged supergravity are metrics with constant scalar curvature  $R = -12g^2$ . Using this, a straightforward calculation then gives for the total (bulk plus boundary) gravity part a finite result, after sending the cut-off  $r = R \rightarrow \infty$ . Namely, after setting  $g = 1$  we get

$$\begin{aligned} I_{\text{tot}}^{\text{grav}} &= I_{\text{bulk}}^{\text{grav}} + I_{\text{bdry}}^{\text{grav}} + I_{\text{ct}}^{\text{grav}} = \frac{(1 - 12s^2)\pi}{32G_4s^2} \\ &\quad - \frac{\sqrt{4s^2 - 1}(1 + 4s^2(-3 + 8s^2))\pi}{16G_4s} , \end{aligned} \quad (4.28)$$

where we note that the contribution on the second line comes entirely from the boundary terms. Although the gauge field is not (anti)-self-dual, it is straightforward to compute its on-shell action, which is still finite, namely we get

$$I_{\text{bulk}}^F = \frac{(1 + 4s^2)\pi}{32G_4s^2} - \frac{\sqrt{4s^2 - 1}(1 + 4s^2(1 - 8s^2))\pi}{16G_4s} . \quad (4.29)$$

Therefore for the total on-shell action we obtain

$$I = \frac{\pi}{2G_4} + (s^2 - \frac{1}{4})^{3/2} \frac{\pi}{G_4s} . \quad (4.30)$$

Notice this makes sense for any  $s > 1/2$ . Moreover, in the  $s \rightarrow 1/2$  limit the second term vanishes and we are left with a result that is the same as that for the round three-sphere  $S^3$ . This might seem a contradiction, but in fact if we look back at where this result comes from, we see that in this limit

$$\lim_{s \rightarrow 1/2} I_{\text{tot}}^{\text{grav}} = \frac{\pi}{4G_4} , \quad (4.31)$$

which is the correct contribution expected from the (singular)  $\text{AdS}_4/\mathbb{Z}_2$  solution with round  $S^3/\mathbb{Z}_2$  boundary. However, we get an equal non-zero contribution from the gauge field action

$$\lim_{s \rightarrow 1/2} I_{\text{bulk}}^F = \frac{\pi}{4G_4} , \quad (4.32)$$

despite the fact that the gauge field curvature  $F \rightarrow 0$  in this limit. The calculation captures correctly the contribution from the flat torsion gauge field, which indeed cannot be turned off continuously since in the bulk has one unit of flux through the vanishing  $S^2$  at the  $\mathbb{Z}_2$  singularity. More precisely, the complement of the singular point has topology  $\mathbb{R}_+ \times S^3/\mathbb{Z}_2$ , and the gauge field is a flat connection on the non-trivial torsion line bundle over this.

## 5 Discussion

In this letter we have extended the results of [2], discussing a new class of supersymmetric solutions of  $d = 4$ ,  $\mathcal{N} = 2$  gauged supergravity, which in turn uplift to solutions of eleven-dimensional supergravity. The solutions in section 3 provide the holographic duals to  $\mathcal{N} = 2$  supersymmetric gauge theories on the background of a biaxially squashed three-sphere and a  $U(1)$  gauge field, whose localized partition function was recently computed in [3]. In particular, as in [2], we have shown that the bulk metric, gauge field, and Killing spinors reduce precisely to their field theory counterparts on the boundary. Moreover, the holographic free energy is identical to the leading large  $N$  contribution to the field theoretic free energy computed from the quiver matrix model. The solution is a special case of the general class of supersymmetric Plebanski-Demianski solutions [5], but it differs from the solution discussed in [2] in various respects. The graviphoton field is again an instanton, hence the bulk metric is Einstein, but it is not now diffeomorphic to  $\text{AdS}_4$ . The results of [2], and of this letter, suggest that the AdS/CFT correspondence is a useful setting for studying supersymmetric gauge theories on curved backgrounds.

We conclude noting that although the results presented here share a number of similarities with those in [6, 9], there are some crucial differences that are worth summarizing. In contrast to the solutions we have discussed, the AdS-Taub-NUT and AdS-Taub-Bolt solutions in [6, 9] are *not* supersymmetric, and moreover no gauge field was turned on. In addition, while those solutions have the same biaxially squashed  $S^3$  boundary, the boundary of our Eguchi-Hanson-AdS solution has the different topology  $S^3/\mathbb{Z}_2$ . Therefore, although we have computed the free energy for both families, it does not make sense to compare them along the lines of [6, 9]. It would be very interesting to understand the precise field theory dual interpretation of the Eguchi-Hanson-AdS solution discussed here.

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